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ALGEBRAIC SURFACES, THEIR CYCLES AND INTEGRALS.

By S. Lefschetz.

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Introduction.

The theory to be considered here has been created by Emile Picard.[*] There are two important Mémoires due to Poincaré bearing largely on the topological phases of it.[†] A short but very interesting note by J. W. Alexander[‡] and one by the writer[§] practically make up the bibliography of the question.

It seems a reasonable requirement that purely topological properties should be derived by direct topological methods, and this requirement we have endeavored to meet here. In this we have proceeded along the lines of Poincaré's second Mémoire simplifying his proofs and completing his results at many important points. This has been followed by the transcendental theory wherein Picard's work on double integrals of the second kind has been much simplified: (*a*) By making their reduction to a simpler form rest upon a preliminary treatment of periodicity. (*b*) By developing their theory without making use of integrals of total differentials of the third kind. In fact reversing Picard's order we have shown that the properties of these integrals and of his number d may be deduced from those of double integrals. The paper concludes with a summary of Severi's related theory of the base and a rapid mention of some allied recent developments, in part unpublished, due to the writer.

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[*] Its exposition forms the major part of the Picard-Simart Traité des fonctions algébriques de deux variables. Paris, Gauthiers-Villars, 2 vols.

[†] Sur les cycles des surfaces algébriques, Jour. de Maths., ser. 5, vol. 8 (1902), pp. 169–214. Sur les périodes des intégrales doubles, ibid., ser. 6, vol. 2 (1906), pp. 135–189. These papers will be called in the sequel, "first Mémoire" and "second Mémoire."

[‡] Sur les cycles des surfaces algébriques . . . , Rendiconti dei Lincei, ser. 5, vol. 23 (1914), pp. 55–62.

[§] Sur certains cycles à deux dimensions des surfaces algébriques, Rendiconti dei Lincei, ser. 5, vol. 26 (1917), pp. 228–234.

PRELIMINARY NOTIONS.

§ 1. *Manifolds and their cycles.**

1. Given a system of real equations with real variables

$$x_i = \theta_i(u_1, u_2, \cdots, u_n); \qquad (i = 1, 2, \cdots, n' > n),$$

where in the range of variation of the (u)'s the (θ)'s are continuous, analytical, and with functional determinants nowhere all zero, we may obtain by analytical continuation a new system and so on. To some of these systems may be adjoined certain conditional inequalities $\varphi(u_1, u_2, \cdots, u_n) > 0$. The point set thus defined constitutes an n-dimensional *manifold* M_n in n-space. By changing one of the inequalities into an equation we obtain a *boundary* of M_n and it will be assumed that the boundaries are grouped into $(n - 1)$-dimensional manifolds $M^1{}_{n-1}, M^2{}_{n-1}, \cdots, M^s{}_{n-1}$.

The familiar notion of the two directions assignable to a line segment may be extended to a certain class of manifolds, those said to be "two-sided." If a small directed circuit drawn on an M_2 can be displaced, then returned to its first position with changed sense, M_2 is said to be *one-sided.* If the sense is invariant M_2 is said to be *two-sided,* and each sense may be considered as defining a *sense* of M_2, so that when we invert the circuit we agree to consider M_2 as replaced by $- M_2$.

When M_2 is thus sensed its boundaries will also be if we make the convention that a bounding line is to be so sensed that when the above directed circuit comes into contact with it there is agreement in the senses impressed by circuit and bounding line upon the small arc common to them. All this can be readily extended to an M_n. *Only two-sided manifolds will be considered here.*

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* For a more complete discussion along the same line see Poincaré, Jour. de l'Ec. Polyt., ser. 2, vol. 1 (1895), § 3; Picard-Simart, vol. 1, p. 20; Heegard, Bull. Soc. de Math., vol. 44 (1916), p. 197. For a more formal treatment better adapted to questions of pure analysis situs see the Dehn-Heegard article on the subject in the Enzyk. der math. Wiss., III, AB3, also a paper by Veblen and Alexander in the Annals of Math., ser. 2, vol. 14 (1913), p. 163.

The relation between M_n and its sensed boundaries will be expressed by a congruence $M_n \equiv \Sigma M^i{}_{n-1}$, or $M_n \equiv 0$ when M_n is *closed* (without boundaries). If we merely wish to indicate that the (M_{n-1})'s bound some M_n in an $M_{n'}$ we shall write a *homology* $\Sigma M^i{}_{n-1} \sim 0$ (mod $M_{n'}$), or simply $\Sigma M^i{}_{n-1} \sim 0$, when no doubt exists as to the $M_{n'}$ in question.

The term " manifold " will be extended to cover an algebraic sum of manifolds as just defined, the boundary being then defined as the algebraic sum of the boundaries. When the (M_n)'s are in an $M_{n'}$, $\Sigma\lambda_i M_n{}^i$, (λ_i integer) may be interpreted as an aggregate of λ_1 manifolds very near $M_{n'}$ in $M_{n'}$, of λ_2 manifolds very near $M_n{}^2$ in $M_{n'}$, etc., the manifolds being all united into one by n-dimensional " tubes " (pairs of opposite segments if $n = 1$, ordinary tubes if $n = 2$, etc.). With these definitions we are enabled to combine congruences and homologies by addition after multiplying them by arbitrary integers. A manifold in the extended sense may now be " connected " without necessarily behaving in the neighborhood of every point not on the boundary like the interior of an n-dimensional sphere.

A closed M_k (of the extended type) contained in an M_n (of the restricted type) is called a *k-dimensional cycle* of M_n (linear cycle if $k = 1$, zero-cycle if $M_k \sim 0$ (mod M_n)).

Cycles are independent if they satisfy no homology. The number R_k of independent k-dimensional cycles of M_n is called its kth *index of connectivity* (linear index if $k = 1$). The numbers $1 + R_k$ have also been called Betti numbers by Poincaré who proved most of their known properties and in particular showed that $R_k = R_{n-k}$.*

We have already mentioned displacements. Only such displacements of an M_n will be considered during which it generates an M_{n+1}. If M_n is displaced in an $M_{n'}$, its extreme positions are equivalent mod. $M_{n'}$, for they form the boundary of the manifold generated during the motion.

Remark.—It has been tacitly assumed that all manifolds considered are finite. We may introduce infinite manifolds by means of systems,

$$\frac{1}{x_i} = \theta_i(u_1, u_2, \cdots, u_n),$$

where the (θ)'s have the same properties as previously.

2. On a closed sensed M_2 of the restricted type we consider the two linear cycles α, β, concerning which we assume that they have only a finite number of intersections simple for each cycle and at which they are

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* Complément à l'Analyse Situs, Rendiconti del Circ. Mat. di Palermo, vol. 13 (1899).

not tangent. If these conditions are not fulfilled we deform one of the cycles until they are. From one of the points of intersection O we take a small segment OA on α in its positive direction and a similar segment OB on β and we join them by a small arc AB of M_2 not cut in two by one of the cycles. Finally we sense OAB so that its vertices are met in the order named. This being done for all intersections there will be some, n in number, for which the triangle is a positive circuit of M_2 and others, n' in number, for which it is just the opposite. The difference $n - n'$ will be designated by (α, β). This character was first introduced by Poincaré,* who proved its properties and made extensive use of it in his investigations. Among these properties the following are easily verified.

$$(a)\ (\alpha, \beta) = -(-\alpha, \beta) = -(\beta, \alpha).$$

$$(b)\ (\alpha + \beta, \gamma) = (\alpha, \gamma) + (\beta, \gamma).$$

$$(c)\ \text{If } \alpha \text{ or } \beta \text{ are } \sim 0, \text{ then } (\alpha, \beta) = 0.$$

A similar character to be designated by (M_p, M_{n-p}) may at once be established for two cycles M_p, M_{n-p} of a closed M_n of the restricted type, (*a*) and (*b*) following then immediately. It is then easy to show that (*d*) if M_q, $q > p$ is closed of the restricted type and goes through M_p intersecting M_{n-p} in an M_{q-p} then (M_p, M_{q-p}) computed relatively to M_q is equal to $\pm (M_p, M_{n-p})$ computed relatively to M_n. By means of (*d*) we can then extend (*c*). Properties (*b*), (*c*) show that (M_p, M_{n-p}) is really a simultaneous invariant of the two cycles with respect to the operation of "homology."

If $p = n - 1$ we have in addition this: When M_{n-1} is not ~ 0, there is a cycle M_1 such that $(M_p, M_1) = +1$. It is the cycle known to exist which meets M_p at one point only, or else its opposite. Hence M_{n-1} is not a zero cycle if and only if there is a linear cycle M_1 such that $(M_n, M_1) \neq 0$.

3. We shall have occasion to consider simple and double integrals. Nothing need be said about the former.

By

$$\sum_{j,k} \int \int_{M_2} R_{jk}(x_1, x_2, \cdots, x_{n'}) dx_j dx_k, \qquad (M_2 \text{ finite})$$

is meant this: M_2 can be subdivided into a finite number of domains M_2^i within which the (x)'s are analytical in two variables u_1, u_2, their jacobians being nowhere all zero, and it is assumed that the (R)'s are analytical in the same conditions. Denoting by $\overline{M}_2^i$ the image of M_2^i in the u_1u_2 plane the double integral is by definition equal to

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* Jour. Ec. Polyt., § 9. He uses the notation $N(\alpha, \beta)$.

$$\sum_{i,j,k}\int\int_{\overline{M}_2{}^i} R_{jk}\frac{D(x_j,\, x_k)}{D(u_1,\, u_2)}\,du_1du_2.$$

The double integrals in the sum can be reduced to line integrals extended over the boundaries of the $(\overline{M}_2)$'s and the directions in which these line integrals must be taken on the boundaries are determined by the order in which u_1 and u_2 are taken. Thus this order determines a direction on the boundaries hence also on those of the manifolds $M_2{}^i$ and finally a sense on M_2. Conversely if we assign a sense on M_2, we determine the order in which u_1 and u_2 must be taken and therefore the sign of the integral. Thus there is a definite meaning to be attached to an integral over a sensed M_2. From the manifold unsensed we may obtain the absolute value of the integral and once the sense is known we may also obtain the sign. This is obviously similar to the case of simple integrals and may be extended to integrals of any multiplicity.

§ 2. *Algebraic curves and their integrals.*

4. A plane algebraic curve C of order m, is the locus of the points whose real or complex coördinates satisfy an equation $f(x, y) = 0$, where $f(x, y)$ is a polynomial of order m. The curve is said to be irreducible if f itself is. The function $y(x)$ thus defined is called an *algebraic function* of x.

In general C has multiple points but it is always reducible to a curve having only double points with distinct tangents for singularities. This reduction can be effected by a *birational transformation* that is, by a transformation such that the coördinates of a point on one curve are rational in those of the corresponding point on the other and vice versa. We shall then assume that C has only these simple singularities and moreover that it lies in general position with respect to the axes and the line at infinity.

A Riemann surface is defined as an M_2 whose points are in point to point correspondence without exception with the real and complex points of C. We may obtain one thus: Let

$$x = x' + ix'', \qquad y = y' + iy'',$$

$$f(x, y) = P(x', x'', y', y'') + iQ(x', x'', y', y''),$$

where the new quantities introduced are all real. The M_2 represented in the real four-space of the variables x', x'', y', y'' by $P = Q = 0$ answers the question.

The Riemann surface is a closed two-sided manifold. It is obviously closed and that it is two-sided may be shown thus: The infinite and branch points are ordinary points, hence a displacement of a sensed circuit ζ returning it to its original position with sense inverted can be assumed such that ζ never crosses these points. Were such a displacement possible it would also be possible to invert by a finite displacement the circuit ζ' projection of ζ in the plane $y' = y'' = 0$ (complex x plane) which is not the case.

The linear index R_1 of the Riemann surface is even and given by

$$R_1 = (m - 1)(m - 2) - 2\delta = 2p,$$

where δ is the member of double points and p is the *genus*. Both R_1 and p are invariants under birational transformations.

There are $2p + m - 1$ linear cycles of which no combination bounds a finite part of the surface.

5. By *abelian integral* belonging to C is meant one of the type

$$\int R(x, y)dx, \tag{1}$$

where $R(x, y)$ is a rational point-function on C. This integral can have no other singularities than poles or logarithmic points in finite number. The integral is said to be of the *first kind* if it is finite everywhere and it is said to be of the *second* or *third* kind according as it has or has not poles for only singularities.

The value of (1) taken over a path with fixed extremities is unchanged when the path does not cross a logarithmic point. Any other path with the same extremities leads to a value differing from the first by a *period*, that is by the value of (1) taken over a linear cycle. The period is *logarithmic* if the cycle is merely a circuit surrounding a logarithmic point, *cyclic* if otherwise. *The sum of the logarithmic periods is equal to zero.*

An abelian integral (1) without logarithmic points at finite distance can be reduced by subtraction of a rational fraction $S(x, y)$ to the type

$$\int \frac{Q(x, y)dx}{f_y'}, \tag{2}$$

where Q is a polynomial *adjoint* to C, that is vanishing at its double points. $S(x, y)$ may be determined by operations rational with respect to the coefficients of R and f.

The integral (2) has $2p + m - 1$ distinct periods and by properly choosing Q these may be made to assume arbitrary values. They correspond to as many cycles of which no combination bounds a finite part of the Riemann surface.

If (1) is without periods it is equal to a rational fraction which may be rationally determined in the same sense as above.

TOPOLOGY OF ALGEBRAIC SURFACES.

§ 1. *Generalities.*

6. The definition of algebraic surfaces is the same as that given for algebraic curves. We shall consider throughout a surface F of order m, equation $F(x, y, z) = 0$, and we assume it to be irreducible, in general position with respect to the axes and the plane at infinity and to have no other singularities than a double curve, a finite number of triple points triple also for the curve, and of pinch-points, or points of the double curve where the two tangent planes coincide. It may be proved that any algebraic surface is birationally transformable into one having no other singularities than these.*

An algebraic surface intersects F in a finite number of space algebraic curves or curves of which the projections on any plane are plane algebraic curves. A linear system of surfaces cuts out in F a linear system of curves and if C is the generic curve of the system, the latter is usually denoted by $|C|$. The system of the plane sections will be denoted by $|H|$ and the pencils cut out by the planes $x =$ Ct., $y =$ Ct., will be denoted by $\{H_x\}$, $\{H_y\}$.

It may be stated here that with some slight changes in wording $|H|$ could be everywhere replaced by a linear system $|E|$ of dimensionality $s \geqq 2$, not containing a subsystem of ∞^{s-2} reducible curves,† and such that there is no simple point of F multiple for all the curves E through it.

The surface F may be represented by an M_4 belonging to a real six-space, and often called its Riemann image. Within this M_4 the algebraic curves of F are represented by (M_2)'s and in the sequel no distinction will be made between curves and surfaces and their representative manifolds. That the M_4 is two-sided may be proved as for Riemann surfaces.

§ 2. *Linear and three-dimensional cycles.*

7. An arbitrary linear cycle Γ_1 of F may be composed of several distinct circuits, but if so we may join them by pairs of opposite segments so as to obtain a cycle $\sim \Gamma_1$ and composed of a single circuit. Let us suppose then that Γ_1, already possesses this property.

Let A be a base point of $\{H_x\}$. We may add to Γ_1 a pair of opposite segments going from A to a point on the cycle, which, still possessing the above property, will now go through A. Let B be an arbitrary point of the cycle and in its H_x join A to B by an arc AB. As B starts from A and describes Γ_1, AB first reduced to the point A varies continuously in

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* Beppo Levi, Annali di mat., ser. 2, vol. 26.

† The necessity of this restriction is explained in the course of the proof of a certain fundamental theorem (No. 11).

F and when B returns to A it reduces to a closed circuit Γ_1', contained in a certain curve H_{x_0}. $\therefore \Gamma_1 \sim \Gamma_1'$ since they bound the M_2 swept out by the arc AB. For a similar reason if we displace H_{x_0} from its position to that of any other H, Γ_1' will take in H a position $\Gamma_1'' \sim \Gamma_1'$, hence $\Gamma_1'' \sim \Gamma_1' \sim \Gamma_1$ and finally we see that *any linear cycle is homologous to one contained in an arbitrary plane section.*[*]

Remark. Obviously this proof holds if $\{H_x\}$ be replaced by any linear pencil having at least *one* base point (pencil of positive degree). Any linear cycle is homologous to one contained in an arbitrary curve of the pencil.

8. Let us draw cross cuts from one point y_0 of the y plane to the critical points $a_1, a_2, \cdots, a_N$, or points for which the plane $y =$ Ct., is tangent to the surface whose class is therefore N. It is only when y is a critical value that H_y may lose cycles for then and then only its genus decreases. Hence a cycle which is returned to a homologous position when y turns around any critical point is also returned to such a position when y describes any closed circuit, and for this reason will be called *invariant.* When y describes its plane without crossing the cuts, an invariant cycle Γ_1 generates a three-dimensional manifold Γ_3 bounded by the locus of Γ_1, when y describes the lips of the cuts. But owing to the invariance of Γ_1, this boundary is composed of a sum of mutually opposite parts, hence Γ_3 is a three-dimensional cycle. Conversely such a cycle Γ_3 intersects H_y into an invariant linear cycle Γ_1.

If Γ_3 *forms boundary on* F *so does* Γ_1 *on* H_y (*and a fortiori on* F) *and conversely if* Γ_1 *forms boundary on* F, *such is also the case for* Γ_3. For first if Γ_1 bounds an M_2 as y describes its plane this M_2 will sweep an M_4 bounded by Γ_3. Next let Γ_1' be any linear cycle of F, Γ_1'' the homologous cycle in H_y. We have $(\Gamma_3, \Gamma_1') = (\Gamma_3, \Gamma_1'')$, $(\Gamma_3, \Gamma_1'') = \pm (\Gamma_1, \Gamma_1'')$, (No. 2). Now if $\Gamma_3 \sim 0$, $(\Gamma_3, \Gamma_1') = 0$ whatever Γ_1' and therefore also $(\Gamma_1, \Gamma_1'') = 0$, and as Γ_1'' is after all a perfectly arbitrary cycle of H_y, $\Gamma_1 \sim 0$ mod H_y and a fortiori mod F.

Remark. We have not shown that any non-zero cycle Γ_1 of F contained in an H_y is the trace of a Γ_3, but only that this is the case for invariant cycles. In point of fact it is shown below that only a certain multiple of Γ_1 is homologous to a trace of Γ_3, mod F.

9. Any H_y contains a cycle Γ_1 homologous to an arbitrary one of F and composed of a single circuit at all of whose points the arc is analytical. Γ_1 may have multiple points but we may assume that the branches through them have distinct tangents as this condition can always be fulfilled by slightly deforming Γ_1. The curves of $\{H_x\}$ which meet Γ_1

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[*] Compare with Picard-Simart, vol. I, p. 85, also Poincaré, 1st Mémoire, § 4, p. 200.

determine a three-dimensional cycle Γ_3 behaving everywhere like an M_3 in the restricted sense except perhaps at the base points of the pencil and also along a certain multiple manifold. However the various branches of Γ_3 through an arbitrary point of this manifold have again the indicated behavior. H_y intersects Γ_3 into several circuits besides Γ_1, say Γ_1', Γ_1'', $\cdots$, $\Gamma_1^{(n-1)}$, with $n \leqq m$, for on each circuit must lie at least one of the m intersections of H_y with an H_x of Γ_3. As y varies the circuits vary also, but two of them $\Gamma_1^{(i)}$, $\Gamma_1^{(j)}$ *can never coalesce into one.* For were this to happen for $y = y_0$ as y would tend towards y_0 the complete intersection of H_y and Γ_3 would acquire a multiple point A which would disappear when y would leave y_0 along at least one path through it. But the H_x through A must then be part of the multiple manifold of Γ_3 and therefore there must be on the intersection in question a multiple point tending towards A when y approaches y_0 along any path whatever.

Now since Γ_3 is a connected manifold we can always make y describe a closed circuit so chosen that $\Gamma_1^{(i)}$ is returned to H_y in the position of $\Gamma_1^{(j)}$ hence $\Gamma_1^{(i)} \sim \Gamma_1^{(j)}$, that is Γ_1, Γ_1', $\cdots$, $\Gamma_1^{(n-1)}$ are homologous cycles of F. It follows that Γ_3 intersects H_y into a linear cycle obviously invariant, $\sim n\Gamma_1$. *Hence some multiple of any cycle of F is homologous to an invariant cycle, mod. F.*

It will be observed again that if Γ_1 is in H_y, $n\Gamma_1$ may be homologous to an invariant cycle, mod. F, though not mod. H_y.

It follows from the above that the number of independent invariant cycles, is the same as the number R_1 of independent linear cycles of F. But from what we have seen in No. 7 follows that the number of independent invariant cycles is equal to the number R_3 of independent three-dimensional cycles. Hence $R_1 = R_3$, which is Poincaré's theorem for the special case of algebraic surfaces.

§ 3. *Proof that the index R_1 is even. Irregularity of the Surface.*

10. The property to be derived presently is usually obtained by analytical methods.* Our treatment here will be wholly topological.

Let us draw on H_y as many independent invariant cycles as we may, say Γ_1^1, Γ_1^2, $\cdots$, Γ_1^q such that $(\Gamma_1^i, \Gamma_1^j) = 0$, $(i \neq j, i, j = 1, 2, \cdots, q)$. Since they are independent cycles of H_y we may find q other cycles of it, say Γ_1^{q+1}, Γ_1^{q+2}, $\cdots$, Γ_1^{2q}, such that Γ_1^{q+1} meets Γ_1^i, but not Γ_1^j, $i \neq j < q$. Assuming these new cycles properly sensed we will have

$$(\Gamma_1^i, \Gamma_1^{q+1}) = +1, \qquad (\Gamma_1^i, \Gamma_1^{q+j}) = 0, \qquad i \neq j.$$

I say now *that the $2q$ cycles Γ_1^i, Γ_1^{q+i} are independent cycles of F.* For assume a relation

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* See for example Picard-Simart, vol. II, p. 423. The linear index is denoted there by r.

$$\sum_{i=1}^{q}\lambda_i\Gamma_1{}^i + \sum_{i=1}^{q}\mu_i\Gamma_1{}^{q+i} \sim 0$$

and let $\Gamma_3{}^i$, $i \leqq q$, be the three-dimensional cycle of which the invariant cycle $\Gamma_1{}^i$, is the trace on H_y. We have

$$\sum\lambda_i(\Gamma_3{}^j, \Gamma_1{}^i) + \sum\mu_i(\Gamma_3{}^j, \Gamma_1{}^{q+i})$$
$$= \sum\lambda_i(\Gamma_1{}^j, \Gamma_1{}^i) + \sum\mu_i(\Gamma_1{}^j, \Gamma_1{}^{q+i}) = \mu_j = 0 \qquad (j = 1, 2, \cdots, q);$$
$$\therefore \sum\lambda_i\Gamma_1{}^i \sim 0, \qquad (\text{mod. } F),$$

and since the cycles at the right are independent mod. F, it must be that the (λ)'s are all zero which proves our statement.

Consider now any linear cycle Γ_1 of F assumed in H_y and let

$$(\Gamma_1{}^i, \Gamma_1) = \nu_i, \qquad i \leqq q.$$

The cycle $\overline{\Gamma}_1 = \Gamma_1 - \Sigma\nu_i\Gamma_1{}^{q+i}$ will satisfy the relations $(\Gamma_1{}^i, \overline{\Gamma}_1) = 0$, $i \leqq q$. By making y describe properly chosen closed paths, $\overline{\Gamma}_1$ will be returned to certain cycles $\overline{\Gamma}_1, \overline{\Gamma}_1{}^2, \cdots, \overline{\Gamma}_1{}^{(n-1)}$ all $\sim \overline{\Gamma}_1$, mod. F and such that

$$\overline{\overline{\Gamma}}_1 = \overline{\Gamma}_1 + \overline{\Gamma}_1{}^1 + \cdots + \overline{\Gamma}_1{}^{n-1}$$

is an invariant cycle. We have

$$(\Gamma_1{}^i, \overline{\Gamma}_1{}^j) = (\Gamma_1{}^i, \overline{\Gamma}_1) = 0, \qquad \therefore (\Gamma_1{}^i, \overline{\overline{\Gamma}}_1) = 0, \qquad (i \leqq q).$$

It follows from this that $\overline{\overline{\Gamma}}_1$ must be dependent upon the cycles $\Gamma_1{}^i$, $i \leqq q$ for otherwise there would be $q + 1$ independent invariant cycles satisfying the relations

$$(\Gamma_1{}^i, \Gamma_1{}^j) = 0, \qquad (i, j \leqq q + 1),$$

while we assumed that there were at most q. Hence finally *all linear cycles of F are dependent upon the cycles* $\Gamma_1{}^i$, $\Gamma_1{}^{q+i}$, $i \leqq q$, and as these are independent it follows that $R_1 = 2q$ *that is* R_1 *is even* as was to be proved.

The integer q is called the *irregularity* of F. It is its most important invariant with respect to birational transformations.

§ 4. *Linear cycles of the plane sections.*

11. We propose to examine the mode of variation of the linear cycles of H_y as y varies. We have already stated (no. 7) that it suffices to examine what happens near the critical points. Consider for the moment H_y as the classical m-sheeted Riemann-surface image of $z(x)$ when y is fixed, and let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be its branch points. When y tends towards a_i two of these uniting the same two sheets, say α_1, α_2, tend to coincide

and the cycle δ_i surrounding the two points in one of the sheets is certainly invariant when y turns around a_i. Let A, B be two points in the Riemann surface belonging each to one of the sheets united in α_1, α_2. There are two possibilities accordingly as there is or is not a path going from A to B without turning around either α_1 or α_2 alone.

(a) In the first case the path in question followed by going from B to the vicinity of α_1, rotating around it once and returning to A in its sheet yields a cycle δ_i' other than δ_i. *This new cycle is increased by* δ_i *when* y *turns around* a_i. If α_2 had taken the place of α_1 we would have in place of δ_i' the cycle $\delta_i' - \delta_i$ of similar behavior, from which we can easily infer that any cycle when increased by a proper multiple of δ_i' is unchanged when y turns around a_i. Hence in the same conditions any cycle is increased by a multiple of δ_i.

(b) In the second case, for $y = a_i$, the critical points α_1, α_2 will merge into a double point and the two sheets of the Riemann surface will cease to be united there. For $y = a_i$ it will be impossible to go from A to B, that is the curve H_{a_i} is *reducible*. As the pencil $\{H_y\}$ is arbitrary in $|H|$ any similar pencil must contain reducible curves, hence there are ∞^2 reducible plane sections. According to Castelnuovo* F is then a *ruled* or a *Steiner surface* (surface of order four with three concurrent but not coplanar double straight lines). In case of a ruled surface the reducible sections are those by planes through the generators, while for the Steiner surface they are those by the tangent planes, the sections being then composed of two conics.

We can thus state the following fundamental theorem: *To each critical point* a_i *corresponds a cycle* δ_i *of* H_y *invariant in its vicinity, and such that when* y *turns around* a_i *any other cycle is increased by a multiple of it.*

Remark. This theorem holds when $|H|$ is replaced by a system $|E|$ such as described in No. 6. For the Steiner or ruled surface we may take for example $|E| = |2H|$.

§ 5. *Two-dimensional cycles.*†

12. An arbitrary two-dimensional cycle Γ meets an arbitrary algebraic curve and in particular H_y in a finite number of points only, and certainly this condition will be fulfilled if the cycle is adequately deformed.

When y describes a positive circuit around y_1, the points of intersection of the H_y and Γ describe small circuits which may be positively or negatively related to Γ. Let $\theta_1, \theta_2, \cdots, \theta_n$ be those of the first type, $\theta_1', \theta_2', \cdots, \theta_n'$ those of the second. We have in fact $n - n' = (H_y, \Gamma)$, and we remark that:

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* Rendiconti dei Lincei, ser. 5, vol. 3 (1894), pp. 22–26.

† See Poincaré's second Mémoire.

(*a*) If Γ is replaced by an irreducible algebraic curve C of order μ, $n = \mu$, $n' = 0$ as follows at once from the fact that we can permute any two of the circuits $\eta_1, \eta_2, \cdots, \eta_\mu$, now taking the place of the (θ)'s by making y_1 describe adequate closed paths in the y plane.

(*b*) As y_1 varies $n - n'$ is constant, for since $H_{y_1} \sim H_{y_2}$

$$(H_{y_1}, \Gamma) - (H_{y_2}, \Gamma) = (H_{y_1} - H_{y_2}, \Gamma) = 0.$$

In particular if there is a single H_y that does not meet Γ then $n = n'$.

Now suppose we have both a curve $C = H$ and a cycle Γ such that $n \neq n'$. Then, keeping to the notation used above, we join the points on η_h of C to the points on θ_j and θ_k' of Γ by lines in H_y. Moreover, we denote by Ψ_{hj}, Ψ_{hk}' the tubes which these lines describe when y turns around y_1, and by (H), (Γ) the parts of these manifolds exterior to the (θ)'s and (η)'s. We have

$$\Psi_{hj} \equiv \eta_h - \theta_j, \qquad \Psi_{hk}' \equiv \eta_h - \theta_k', \qquad (H) \equiv \Sigma\eta_h, \qquad (\Gamma) \equiv \Sigma\theta_j - \Sigma\theta_k'.$$

$$\therefore m(\Gamma) - (n - n')(H) + \Sigma\Psi_{hj} - \Sigma\Psi_{hk}' \equiv 0.$$

The manifold at the left is closed, meets nowhere H_y and is reducible by deformation to $m\Gamma - (n - n')H$ cycle to which it is therefore homologous. In the same manner we can arrive at a cycle $\sim m\Gamma - (n - n')H$ meeting nowhere a group of curves H_y.

This reduction applies as well when $n = n'$. But in that case we may join the points of H_y describing θ_j and θ_j' by a line in H_y and obtain a tube Ψ_j resting on θ_j, θ_j'. With the same notations as above then

$$\Psi_j \equiv \theta_j - \theta_j', \qquad (\Gamma) \equiv \Sigma(\theta_j - \theta_j').$$

$$\therefore (\Gamma) - \Sigma\Psi_j \equiv 0,$$

and we have here a cycle $\sim \Gamma$ meeting nowhere H_y. Hence if there is an H_y which Γ does not intersect the cycle may be so deformed as not to meet a given group of (H_y)'s. In particular a finite cycle is reducible by a finite deformation to a cycle which does not meet a given group of (H_y)'s. That the deformation may be finite follows from our proof.

§ 6. *Reduction of two-dimensional cycles to a special type.*

13. When y describes the semi-straight line

$$\text{argt.}\ (y - a_0) = \text{Ct.}, \tag{1}$$

H_y generates a manifold K_y. If there are no critical points between two points y_1, y_2 of (1), the manifolds H_{y_1}, H_{y_2} are reducible to each other by a deformation never introducing any new singularities. This deformation sets up between them a point-to-point correspondence with the points at infinity corresponding to themselves. The point P_2 of H_{y_2} corresponding to a point P_1 of H_{y_1} will be called its projection in H_{y_2} and the locus of P_1 during the deformation its projecting line.

In the sequel it will be found convenient to denote the curves H_y corresponding to the critical points a_i by H_i, and the curves corresponding to $y = a_0, \infty$ by H_0 and H_∞ respectively.

Let Γ be a cycle which does not meet H_0, H_∞, or any of the curves H_i. Any finite cycle can be reduced by finite deformation to one having this property. If we project the points of Γ into H_0 the projecting lines form a finite M_3 bounded by $-\Gamma_1$, by a finite portion (H_0) of H_0, and by a certain manifold P that may be described thus: The manifold K_{a_i} corresponding to the semi-straight line through the critical point a_i intersects Γ in two sets of linear cycles such that for all points of one $|y - a_0| < |a_i - a_0|$, while for all points of the other, $|y - a_0| > |a_i - a_0|$. Let γ_i be the cycles of the second set. Then, the projection on H_0 of a system of closed linear cycles in K_y which tends towards γ_i as K_y tends towards K_{a_i} approaches one of two distinct limiting positions γ_i and γ_i' according as argt. $(y - a_0)$ tends towards $a_i - a_0$ from above or below. The projecting lines in the two cases form manifolds with limiting positions P_i', P_i'', and we have

$$P = \Sigma(P_i'' - P_i').$$

Hence,

$$\Gamma \sim (H) + \Sigma(P_i'' - P_i')$$

where, moreover, the two cycles are reducible to each other by a finite deformation.

Now, let $\bar{\gamma}_i$ be the projection of γ_i into H_{y_1} assumed in K_{a_i} and such that $|y_1 - a_0| > |a_i - a_0|$, then slide γ_i along its projecting lines until it coincides with $\bar{\gamma}_i$. During the motion P_i' and P_i'' are deformed assuming final positions Q_i', Q_i'', hence

$$(P_i'' - P_i') - (Q_i'' - Q_i') \sim 0.$$

Now $Q_i'' - Q_i'$ may be generated by a certain cycle δ of H_y when y starting from a_0 describes a closed path surrounding once a_i, the nature of this path being on the whole immaterial. Denoting by δ_i the same linear cycle of H_y as in § 4, and by Δ_i its two-dimensional locus when y describes the straight path a_0a_i, let us then make y_1 tend towards a_i on the line

$$\text{argt.}\ (y - a_0) = \text{argt.}\ (a_i - a_0).$$

When y describes the closed path around a_i, δ undergoes an increment of $\lambda_i\delta_i$, and when y_1 coincides with a_i, $Q_i'' - Q_i'$ will have become $\lambda_i\Delta_i$ hence

$$Q_i'' - Q_i' - \lambda_i\Delta_i \sim 0; \qquad \therefore \Gamma \sim \Sigma\lambda_i\Delta_i + (H_0).$$

The cycle at the right is at finite distance, hence at once

$$- \Sigma\lambda_i\delta_i \equiv (H_0), \qquad (\text{mod } H_0).$$

Conversely if this last congruence is verified, that is if $\Sigma\lambda_i\delta_i$ bounds a finite portion of H_0, $\Sigma\lambda_i\Delta_i + (H_0)$ is a two-dimensional cycle without infinite points. This is an immediate consequence of the congruences $\Delta_i \equiv \delta_i$. Thus: *A finite cycle is reducible by finite deformation to a sum of multiples of the* (Δ)*'s plus a part of* H_0. *The linear cycle* $\Sigma\lambda_i\delta_i$ *bounds a finite part of* H_y *and conversely to such a cycle corresponds a finite cycle* $\Sigma\lambda_i\Delta_i + (H_0)$.

This very important theorem was discovered by Picard* the proof here given being a modification of one due to Poincaré.†

14. A plane section or in fact any irreducible algebraic curve C forms a two-dimensional cycle. Such a cycle cannot be dependent upon finite cycles. For if Γ is finite $(\Gamma, H) = (\Gamma, H_\infty) = 0$, while on the contrary $(C, H) \neq 0$ (No. 12).

Since H is independent of the finite cycles and on the other hand whatever Γ there is a finite cycle $\sim m\Gamma + nH$, *the number of independent finite cycles is precisely equal to* $R_2 - 1$.

§ 7. *Formula for the index* R_2.

15. The number of independent finite cycles may also be derived in a different manner and this will lead us to the formula for R_2. We have seen that when a finite cycle Γ is reduced to the type $\Sigma\lambda_i\Delta_i + (H_0)$ then $\Sigma\lambda_i\delta_i$ assumed in H_0 bounds a finite part of it, which we express by

$$\Sigma\lambda_i\delta_i \equiv - (H_0) \tag{2}$$

and we must first determine the number of independent congruences (2).

Let γ be any linear cycle of H_0 bounding an M_2 of F. By a discussion similar to that of Nos. 13, 14 we may show that there is an $M_2' = nM_2 + \Gamma_1'$, ($n \neq 0$, Γ' two-dimensional cycle) meeting nowhere the curves H_i, H_∞ and bounded by $n\gamma$. From this will follow a congruence

$$M_2' - \Sigma\lambda_i\Delta_i + (H_0) \equiv 0; \qquad \therefore n\gamma - \Sigma\lambda_i\delta_i \equiv - (H_0),$$

that is if $\gamma \sim 0$ mod. F, there is a finite part of H_0, *bounded by a combination of the cycles* γ, δ_i. The congruences $\delta_i \equiv \Delta_i$ show that the (δ)'s themselves form boundary on F. Hence the number of (δ)'s which do not satisfy a congruence (2) is exactly equal to the number of zero cycles of F contained in H_0 of which no combination forms the boundary of a finite part of H_0, that is finally to $2p - 2q + m - 1$.* As there are N cycles δ_i, the number of distinct congruences (2) is $N-(2p-2q+m-1)$.

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* Picard-Simart, vol. II, p. 335.

† Second Mémoire, p. 155, §§ 4, 5.

16. To each congruence (2) corresponds a two-dimensional cycle

$$\Gamma = \Sigma\lambda_i\Delta_i + (H_0).$$

We have to find how many of these cycles are ~ 0 though corresponding to distinct congruences.

Assume that Γ corresponding to (2) bounds an M_3 and let γ be the linear cycle intersection of M_3 and H_y. M_3 can be considered as the locus of γ when y describes the y plane without crossing the cuts a_0, a_i. The boundaries of M_3 are obtained when y describes the lips of these cuts.

Now the congruence (2) shows that when y describes the lips of the cut a_0a_i the difference of the two manifolds generated by γ is $\lambda_i\Delta_i$ and therefore when y turns around a_i, γ receives the increment $\lambda_i\delta_i$. Hence if the (λ)'s are not all zero, γ *is not an invariant cycle.*

Let Γ' be a tube contained in M_3 and having for axis the position of γ in H_∞. If we suppress in M_3 the portion within Γ' we obtain a finite M_3' bounded by Γ and Γ'. Thus if $\Gamma \sim 0$ *it bounds together with a tube of axis in* H_∞ *a certain finite* M_3.

Conversely let γ be a linear cycle of H_y and consider its locus Γ' when y describes a circle of large radius. $\Gamma' \sim 0$ and moreover it will meet nowhere the curves H_i, H_0, if as we may arrange it γ does not go through a base point of $\{H_y\}$. Let us deform the circle until it coincides with the lips of the cuts a_0a_i. There will be a corresponding finite deformation of Γ' reducing it ultimately to a cycle $\Gamma = \Sigma\lambda_i\Delta_i + (H_0)$. $\lambda_i\Delta_i$ indicates the difference between the manifold generated by γ on both sides of a_0a_i and therefore $\lambda_i\delta_i$ is the increment of γ when y turns around a_i. Hence if γ is not an invariant cycle the (λ)'s are not all zero and we have here a finite two-dimensional zero cycle relative to a congruence (2) with coefficients λ_i not all zero. The number of distinct congruences corresponding to such cycles is therefore equal to the number of distinct cycles of H_y of which no combination is invariant, that is to $2p - 2q$.

17. It follows from the preceding discussion that there are

$$N - (4p - 4q + m - 1)$$

distinct finite cycles. Equating this to the number $R_2 - 1$ already found we obtain

$$R_2 = N - (4p + m) + 4q + 2.$$

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* Poincaré assumed this number to be $2p + m - 1$ hence did not arrive at the correct value for R_2. The correct value here given was obtained by Picard by transcendental methods. See Picard-Simart, vol. II, pp. 377, 398.

The number $I = N - (4p + m)$ formed with the characteristic numbers N, m, p of $|H|$ is a numerical invariant which would remain the same if it were computed from the corresponding characters of a linear system such as $|E|$ of No. 6. It is in fact known as the invariant of Zeuthen-Segre. Various geometers (Segre, Castelnuovo and Enriques) have shown how it may be computed from the characters of any linear pencil of curves. Be it as it may we have the fundamental formula

$$R_2 = I + 4q + 2,$$

in which enter only the invariants of F. It was first correctly derived by Alexander (loc. cit.) by two distinct and very original methods based essentially on a subdivision of F into a generalized polyhedron, much as is done by Poincaré in his first Mémoire, followed by an application of the Euler-Poincaré polyhedron formula.

INTEGRALS BELONGING TO ALGEBRAIC SURFACES.

§ 1. *Double integrals and their residues.*

19. Given a rational point function $R(x, y, z)$ on F we may set up a double integral

$$\int\int R(x, y, z)dxdy. \tag{1}$$

This integral is assumed taken over an M_2 of the surface. Just as with abelian integrals we are here led to integrate over cycles at all points of which the integral is finite. The corresponding value of (1) is called a *period.* A cycle may be reducible by deformation to a line and yet yield a period. It is then a tubular manifold having for axis this line which must be situated in the curve of discontinuity of (1). The period is then said to be a *residue.*

19. The following theorem will prove useful in the study of residues.* Here and in the sequel we denote throughout by $\partial U/\partial x$, $\partial U/\partial y$, the partial derivatives of $U(x, y, z)$ when z is considered as a function of x, y, reserving the notation U_x', U_y', U_z', for the partial derivatives when the variables are all to be considered as independent. Then, *if at all points of a two-dimensional cycle* Γ, *the rational function* $U(x, y, z)$ *is finite and* $F_z' \neq 0$

$$\int\int_\Gamma \frac{\partial U}{\partial x} dxdy = \int\int_\Gamma \frac{\partial U}{\partial y} dxdy = 0.$$

Let us take for example the first integral. At all points of Γ

$$\frac{D(x, y)}{D(U, y)} = \frac{F_z'}{U_x'F_z' - U_z'F_x'} \neq 0.$$

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[*] It was first given in the writer's Mémoire in the Annali di Mat., ser. III, vol. 36 (1917), § 8.

Hence the change of variables $X = U(x, y, z)$, $Y = y$ transforms Γ into a finite cycle Γ' of the real four-space image of the values of X, Y. Applying Poincaré's extension of Cauchy's fundamental theorem to functions of two complex variables we obtain at once

$$\int\int_{\Gamma} \frac{\partial U}{\partial x} dxdy = \int\int_{\Gamma'} dXdY = 0,$$

as was to be proved.

20. A first corollary is that the integrals

$$(2) \qquad \int\int \frac{\partial U}{\partial x} dxdy, \qquad \int\int \frac{\partial U}{\partial y} dxdy,$$

have no residues. Indeed a cycle corresponding to a residue may always be deformed so as to satisfy the conditions of the theorem.

Let now D be a curve of discontinuity of $R(x, y, z)$ other than the curve at infinity and $g(x, y) = 0$ its projection in the xy plane. We may write

$$R(x, y, z) = \frac{A(x, y, z)}{[g(x, y)]^{\alpha}},$$

where $A(x, y, z)$ is a rational function finite on D and α is a positive integer. Assuming as we may if the axes are properly chosen that $g(x, y)$ contains the variable y, we have identically if $\alpha > 1$,

$$\int\int \left(\frac{A}{g^{\alpha}} + \frac{1}{\alpha - 1} \frac{\partial}{\partial y} \frac{A}{g_y' g^{\alpha-1}}\right) dxdy = \int\int \frac{1}{(\alpha - 1) g^{\alpha-1}} \frac{\partial}{\partial y} \left(\frac{A}{g_y'}\right) dxdy.$$

This coupled with the remark made above shows that as far as its residues with respect to D are concerned the integral

$$(1) \qquad \int\int R(x, y, z) dxdy = \int\int \frac{A(x, y, z)}{[g(x, y)]^{\alpha}} dxdy$$

can be replaced by one of the same type with $\alpha - 1$ in place of α. This can be continued until we arrive at an integral of type

$$(3) \qquad \int\int \frac{A(x, y, z)}{g(x, y)} dxdy$$

and with the same residues as the initial one.

Let now Γ be a tube of axis γ in D corresponding to a residue of (1) and so deformed as to have no points at infinity. Passing from the variables of integration x, y, to the variables x, g, and denoting by Γ' the transformed of Γ in the real four-space corresponding to the last pair of variables, the residue of (3) relatively to Γ is equal to the value of

$$\int\int_{\Gamma'} \frac{A(x, y, z)dxdg}{gg_y'}. \tag{4}$$

Now Γ' is a closed tube of axis γ' in $y = 0$. Let P be a point of γ', $x_1' + ix_1''$ its x. The real three-space $x' = x_1'$ of the complex (x, g) space intersects the manifold $g = 0$ in a line δ through P and Γ' in a closed circuit ϵ surrounding δ. We can slide ϵ in its space until it becomes a circuit ϵ' of the plane $x' = x_1'$ and this without its ever meeting δ. Carrying the process out as P describes γ will result in deforming Γ' into a tube Γ'' generated by a circuit ϵ' of the planes $x =$ Ct., when x describes a certain circuit ζ in its plane. As the deformation from Γ' to Γ'' may be made without ever meeting $g = 0$ we have

$$\int\int_{\Gamma'} \frac{Adxdg}{gg_y'} = \int_{\zeta} \frac{Adx}{g_y'} \int_{\epsilon'} \frac{dg}{g} = 2\pi i \int_{\gamma} \frac{Adx}{g_y'}$$

which shows that the *residues of* (3) *are the periods of the abelian integral*

$$2\pi i \int \frac{A(x, y, z)dx}{g_y'}, \qquad g = 0 \tag{5}$$

belonging to the curve D.

The converse of this proposition is almost evident for we can easily build a continuum Γ' corresponding to a period of (5) and obtain from it a tube Γ.

21. Residues may be classified into two kinds according to the type of residues which they constitute for the corresponding integrals. A *point residue* corresponds to a logarithmic period and a *cyclic residue* to a cyclic period. To a point residue relatively to D corresponds a tube of axis drawn around certain points of the curve—points which are either intersections with other discontinuities or else *accidental* singularities, that is singularities such as the point of contact with a tangent plane for the section by it. Hence if the integral has only one curve of discontinuity deprived of accidental singularities it has no point residues. Finally from a well-known property of logarithmic periods of abelian integrals follows that the point residues corresponding to any curve of discontinuity have a zero sum. We may mention that the residues corresponding to a given point of the surface also have a zero sum.*

§ 2. *Double integrals of the second kind.*

22. The classification of abelian integrals on the basis of their discontinuities may be extended to double integrals. We say with Nöther that

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* See the writer's note in the Quarterly Journal, vol. 47 (1917), pp. 333–348.

$$(1) \qquad \int\int R(x, y, z)dxdy$$

is of the *first kind* if it is finite everywhere. It is of the *second kind* if it behaves in the vicinity of any curve of discontinuity like one of type

$$(6) \qquad \int\int\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}\right)dxdy,$$

where U, V are rational in x, y, z.[*] The difference between (1) and (6) being finite in the vicinity of the curve it follows from No. 20 that (1) has no residues and as we shall see this property is characteristic.

Two integrals of the second kind differing only by one of type (6) are said to be *equivalent*. Our fundamental problem consists in *finding the number of inequivalent integrals of the second kind.*

An integral of the second kind is said to be *improper* if of type (6), *proper* if not of this type.

An integral is of the *third kind* if it belongs to neither of the two kinds already mentioned. Our discussion will be wholly confined to integrals of the second kind.

23. We proceed to show that integrals without residues are of the second kind. For if (1) is infinite on a curve D of projection $g(x, y) = 0$ we may by subtracting a suitable integral (6) reduce it to the form

$$\int\int\frac{A(x, y, z)}{g}dxdy.$$

The residues which are the periods of

$$2\pi i\int\frac{A(x, y, z)dx}{g_y'}, \qquad g(x, y) = 0,$$

being all zero, this abelian integral is a rational function $2\pi i\cdot U(x, y)$ on D, hence

$$\begin{aligned}\int\int\left[\frac{A(x, y, z)}{g}-\frac{(U_x'g_y'-U_y'g_x')}{g}\right]dxdy \\ =\int\int\left[\frac{A}{g}-\frac{D\left(\frac{U}{g}, g\right)}{D(x, y)}\right]dxdy \\ =\int\int\frac{A}{g}dxdy-\int\int\frac{\partial}{\partial x}\left(\frac{U}{g}\right)dxdg\end{aligned}$$

is finite in the vicinity of D, which is sufficient to show that (1) behaves like an integral (6) in the neighborhood of D. Obviously then when there are no residues the integral is of the second kind.*

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[*] This definition differs slightly from Picard's, but seems somewhat more convenient. See Picard-Simart, vol. II, p. 160.

24. Let us now prove *that integrals of type* (6) *preserve their form under birational transformations*. Indeed we have

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=\frac{D(U,y)}{D(x,y)}+\frac{D(x,V)}{D(x,y)};$$

$$\frac{D(U,V)}{D(x,y)}=\frac{\partial}{\partial x}\left(U\frac{\partial V}{\partial y}\right)-\frac{\partial}{\partial y}\left(U\frac{\partial V}{\partial x}\right).$$

Hence if x', y', z' are the new variables and U_1, V_1 rational functions of them

$$\int\int\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}\right)dxdy=\int\int\left(\frac{D(U,y)}{D(x',y')}+\frac{D(x,V)}{D(x',y')}\right)dx'dy'$$

$$=\int\int\left(\frac{\partial U_1}{\partial x'}+\frac{\partial V_1}{\partial y'}\right)dx'dy'$$

as we wished to show. It follows in particular that if an integral (1) behaves like an integral of the second kind in the vicinity of a curve D of F not fundamental for the transformation, its transformed will behave similarly in the vicinity of the transformed curve of D on the transformed surface.

§ 3. *On the periods of certain double integrals.*

25. The reduction and enumeration of integrals of the second kind rests upon the study of the periods of a class of double integrals of which the simplest are of type

$$\int\int\frac{Q(x,y,z)}{F_z'}dxdy, \tag{7}$$

where Q is an adjoint polynomial of the surface. Such an integral has a meaning whenever taken over a manifold at finite distance for the integrand is then never infinite to a power greater than one half. Closely associated to (7) is the abelian integral of H_y

$$\int\frac{Q(x,y,z)}{F_z'}dx. \tag{8}$$

Let $\Omega_i(y)$ be the period of (8) with respect to the cycle δ_i of No. 6, unchanged in the vicinity of the critical point a_i. As (7) extended to a finite portion of H_0 is zero, its value extended to a cycle $\sum\lambda_i\Delta_i+(H_0)$ such as considered in Part I, is equal to

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* Picard-Simart, vol. II, p. 203.

$$(9) \qquad \sum \lambda_i \int_{a_0}^{a_i} \Omega_i(y)dy,$$

and the condition $\sum \lambda_i \delta_i \equiv (H_0)$, gives here since a_0 is arbitrary

$$(10) \qquad \sum \lambda_i \Omega_i(y) = 0.$$

26. If we impose upon a polynomial $g(x, z)$ the condition of being adjoint to the curve $F(x, y, z) = 0$ we merely constrain its coefficients to satisfy a certain number of linear equations with coefficients rational in those of F, at least this will certainly be the case for reasons of symmetry if the curve has no other singularities than double points with distinct tangents. Hence the most general polynomial of a given order adjoint to the curve is a linear combination of a certain number of polynomials with coefficients rational in those of the curve, or in the last analysis of a certain number of polynomials adjoint to the surface and of the given degree in x and z alone if not in the three variables x, y, z. As a consequence (No. 5), an arbitrary integral (8) has $2p + m - 1$ distinct periods and we may give ourselves $2p + m - 1$ such integrals with a non-zero period-determinant of order equal to their number.

27. Starting now with the same adjoint polynomial $Q(x, y, z)$ that appears in (7) and with a polynomial in y, $\varphi(y)$, we form the double integral

$$(11) \qquad \int\int \frac{\varphi(y)Q(x, y, z)}{F_z'} dxdy,$$

and we shall show that its periods may take completely arbitrary values. These periods are quantities.

$$\sum_j \lambda_j^h \int_{a_0}^{a_j} \varphi(y)\Omega_j(y)dy, \qquad \sum_j \lambda_j^h \Omega_j(y) = 0.$$

If our affirmation were incorrect there would have to exist a relation such as

$$\sum_j \mu_j \int_{a_0}^{a_j} y^k \Omega_j(y)dy = 0$$

with coefficients μ_j not all zero, satisfied for all integral values of the exponent, k. This would lead at once to

$$\sum_j \mu_j \int_{a_0}^{a_j} \frac{\Omega_j(y)}{y - a} dy = 0$$

for a arbitrary. Assuming then $\mu_j \neq 0$ the increment $2\pi i \mu_j \Omega_j(y)$ of the left-hand side when y describes a closed circuit surrounding a_j would have to be zero. But in that case $\Omega_j(y) = 0$ and as a consequence the cycles of H_y would all be invariant in the vicinity of a_j, (No. 6), which is impossible if F is not ruled or of Steiner and $\{H_y\}$ is arbitrary. We conclude then that the periods of an arbitrary integral (7) may have any set of $N - (2p + m - 1 - R_1)$ values. As we have seen $2p - R_1$ of the corresponding cycles are reducible by finite deformation to tubes with axes in H_∞, manifolds which correspond to residues of the double integral. We can form integrals (7) for which these residues are equal to zero, the other periods having arbitrary values, and these integrals are of the second kind.

28. Nearly all this applies to integrals of the second kind,

$$\int\int \frac{Q(x, y, z)}{\varphi(y)F_z'}\,dxdy, \tag{12}$$

where Q is adjoint to F and the polynomial φ has no critical value for root. The period corresponding to (9) is still

$$\sum \lambda_j \int_{a_0}^{a_j} \Omega_j(y)\,dy,$$

where now Ω_j is the period of

$$\int \frac{Q(x, y, z)}{\varphi(y)F_z'}\,dx \tag{13}$$

relatively to δ_i. The value of this period does not change when the cuts are made to cross a root y_0 of $\varphi(y) = 0$. Indeed when for example a_0a_j is made to cross y_0 the period increases by $2\pi i$ times the residue of $\Omega_j(y)$ with respect to the pole y_0. But this increment must be zero for its value is also the residue of the double integral of the second kind (13) with respect to a tube having for axis the position of δ_j in H_{y_0}. Similarly if $\omega(y)$ is any period of (13), the function

$$\int \omega(y)\,dy$$

is meromorphic in the neighborhood of the roots of $\varphi(y)$, and in particular if $\omega(y)$ is rational the integral is also a rational function of y. All this ceases to be true when either (12) is not of the second kind or $\varphi(y)$ vanishes at a critical point.

Remark. We have just mentioned periods of (13) rational in y. An example is furnished by a period corresponding to an invariant cycle, for it is meromorphic for all values of y.

29. *If the integral* (12) *is without periods with respect to the finite cycles it can be put in the form*[†]

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[*] See Picard-Simart, vol. II, p. 354.

[†] The proof here given is not essentially different from that in Picard-Simart, vol. II, p. 365, for the special case $\varphi \equiv 1$.

$$(14)\qquad \int\int\left(\frac{\partial}{\partial x}\frac{A}{\varphi F_z'}+\frac{\partial}{\partial y}\frac{B}{\varphi F_z'}\right)dxdy$$

(A, B, adjoint polynomials, φ polynomial in y). Let us recall first that as shown in No. 26 we may choose $2p + m - 1$ integrals

$$(15)\qquad \int\frac{Q_j(x, y, z)}{F_z'}dx \quad (j = 1, 2, \cdots, 2p + m - 1),$$

with a non-zero determinant of periods. Let $\gamma_1, \gamma_2, \cdots, \gamma_{2q}$ be $2q$ invariant cycles of H_y independent mod F, and denote by ω_j, ω_{hj} the periods of (13) and (15) with respect to γ_j, and by Ω_{hk} that of (15) with respect to δ_k. We may consider the equations in the (c)'s

$$\int_{a_k}^{y}\Omega_k(y)dy = \sum_{h=1}^{2p+m-1} c_h\Omega_{hk}(y) \qquad (k = 1, 2, \cdots, N);$$

$$\int_{a_0}^{y}\omega_j(y)dy = \sum_{h=1}^{2p+m-1} c_h\omega_{hj}(y) \qquad (j = 1, 2, \cdots, 2q).$$

Since any cycle of H_y together with the (γ)'s and the (δ)'s can always be so combined as to form the boundary of a finite part of H_y the matrix of the coefficients is of rank $2p + m - 1$. The distinct combinations of the right-hand sides identically zero are all derived from the relations

$$\sum_{k=1}^{N}\lambda_k\Omega_{hk}(y) = 0 \quad (\lambda_k \text{ integer}; \ h = 1, 2, \cdots, 2p + m - 1),$$

which are known to exist and there are $N - (2p - 2q + m - 1)$ of them. But when such a relation is satisfied, $\sum\lambda_k\delta_k$ bounds a finite part of H_y and $\sum\lambda_k\Delta_k + (H_0)$ is a two-dimensional cycle without infinite points. The corresponding period of (12) is then zero by assumption, hence

$$\sum\lambda_k\int_{a_0}^{a_k}\Omega_k(y)dy = 0,$$

or since a_0 is arbitrary and can be replaced by y,

$$\sum\lambda_k\int_{a_k}^{y}\Omega_k(y)dy = 0.$$

This shows that the equations in the (c)'s are compatible, and therefore furnish a unique solution. As the left hand sides and the coefficients are meromorphic for arbitrary values of y as well as for the roots of $\varphi(y)$ (No. 28) the same is true for the (c)'s. The only singularities which they might acquire are the critical values. But when y describes a closed circuit around a_j the quantities

$$\int_{a_k}^{y} \Omega_k(y)dy, \qquad \Omega_{hk}(y),$$

receive increments

$$\lambda \int_{a_j}^{y} \Omega_j(y)dy, \qquad \lambda\Omega_{hj}(y),$$

and therefore the system of equations which we are considering remains invariant. This shows that even a critical value is at most a pole of the (c)'s so that these functions are rational in y.

Setting

$$\sum_{h=1}^{2p+m-1} c_h Q_h(x, y, z) = \frac{A(x, y, z)}{\psi(y)},$$

where A and ψ are polynomials, the first adjoint to F, and considering

$$\int \left(\frac{Q(x, y, z)}{\varphi(y)F_z'} - \frac{\partial}{\partial y}\frac{A(x, y, z)}{\psi(y)F_z'} \right) dx$$

we remark that it reduces to a rational function on H_y, hence

$$\frac{Q(x, y, z)}{\varphi(y)F_z'} - \frac{\partial}{\partial y}\frac{A(x, y, z)}{\psi(y)F_z'} = \frac{\partial U}{\partial x},$$

where U is a rational function. As it can have no poles at a finite distance for y fixed,

$$U = \frac{B(x, z)}{\psi(y)F_z'}.$$

Since A, B can be replaced by $A\psi_1$, $B\psi$, and ψ, ψ_1 by their product, our proof is now complete.

§ 4. *Integrals of the second kind. Reduction and enumeration.*

30. Let D be an irreducible algebraic curve in general position on F, $g(x, y) = 0$ its projection,

$$R(x, y, z) = \frac{A(x, y, z)}{B(x, y, z)} \qquad (A, B \text{ polynomials}),$$

a rational function infinite on the curve. The resultant of B and F considered as polynomials in z alone is of the form $g^a \cdot G(x, y)$ with G a polynomial prime to g, and we have

$$g^a \cdot G = C_1 \cdot D_1 + D_2 \cdot F, \qquad (C_1, D_1, D_2, \text{ polynomials}).$$

Hence on F

$$R = \frac{AC_1}{g^a \cdot G} = \frac{A_1}{g^a \cdot G}.$$

On the other hand, breaking into partial fractions,

$$\frac{1}{g^\alpha \cdot G} = \frac{H_1(x, y)}{\varphi(y) \cdot g^\alpha} + \frac{H_2(x, y)}{\varphi(y) \cdot G(x, y)}, \quad (H_1, H_2, \text{polynomials}).$$

Hence in the vicinity of $g(x, y) = 0$, R is infinite like

$$R_1(x, y) = \frac{A_1 H_1}{\varphi(y) \cdot g^\alpha} = \frac{K(x, y, z)}{\varphi(y) \cdot g^\alpha},$$

and as $\varphi(y)$ vanishes only for the (y)'s of the points for which $g = G = 0$, and the curve D is in general position, φ has no critical value for root. We may note in particular that if $g(x, y) = 0$ intersects F into two curves on one of which R is finite, R_1 will be finite on the same curve.

31. Given an arbitrary integral of the second kind with curves of discontinuity in general position,

$$\int\int R(x, y, z) dx dy = \int\int F_z' \cdot R(x, y, z) \frac{dx\, dy}{F_z'} \tag{16}$$

we may apply the first part of the preceding discussion to RF_z' and put the integral in the form

$$\int\int \frac{A(x, y, z) dx dy}{\prod_i [g_i(x, y)]^{\alpha_i} F_z'}, \tag{17}$$

where $g(x, y) = 0$ is the projection of an irreducible curve of discontinuity D_i, the integral being finite on the other curve of F having the same projection. Having effected if necessary a preliminary transformation of coördinates we are at liberty to assume that the (D)'s occupy a general position with respect to the axes.

The double integral behaves in the vicinity of D_i like one of type

$$\int\int \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) dx dy.$$

In so far as the vicinity of D_i is concerned this last integral, according to what we have just seen, may be replaced by another

$$\int\int \left(\frac{\partial}{\partial x} \frac{C_1}{\varphi(y) g_i^{\beta_i}} + \frac{\partial}{\partial y} \frac{C_2}{\varphi(y) g_i^{\beta_i}} \right) dx dy, \tag{18}$$

where C_1, C_2, φ are polynomials, $\varphi(y)$ having no critical value for root. Subtracting from (17) all the integrals (18) corresponding to its various curves of discontinuity we obtain one of form

$$\int\int \frac{A_1(x, y, z) dx dy}{\varphi(y) \prod_i g_i^{\alpha_i}(x, y) F_z'}$$

equivalent to (17) but infinite only at infinity and on the curves $\varphi(y) = 0$, of which none correspond to critical values. Clearly the polynomial $A_1(x, y, z)$ must vanish at least once on the complete intersection of F with $g_i = 0$ if it is a simple curve of F, and at least twice if it is double. The (g)'s being assumed irreducible, we can apply Nöther's generalized theorem* which gives

$$A_1 = Ag_i + BF \qquad (A,\ B \text{ polynomials}).$$

Hence on the surface

$$\frac{A_1}{g_1} = A, \text{ polynomial,}$$

so that the integral may be replaced by one of the same form with $\alpha_i - 1$ in place of α_i. This can be continued indefinitely and for all the (g)'s so that the integral assumes finally the form

$$\int\int \frac{Q(x, y, z)dxdy}{\varphi(y)F_z'} \qquad (\varphi \text{ has no critical value for root}) \tag{19}$$

and as it is finite on the double curve the polynomial $Q(x, y, z)$ is adjoint to the surface.

32. According to No. 27 we may form an integral

$$\int\int \frac{P(x, y, z)dxdy}{F_z'} \qquad (P \text{ adjoint polynomial}) \tag{20}$$

with periods equal to those of (19). Their difference is of the same type as (19) and therefore (No. 29) improper of the second kind, hence (19) and (20) are equivalent. We conclude then from this that *any integral of the second kind is equivalent to one of form* (20) *that is to an integral finite at finite distance.*† Since H_∞ is an arbitrary H we may affirm *that any integral of the second kind is equivalent to one infinite on an arbitrary plane section and nowhere else.* This shows that such an integral behaves as if it were improper in the neighborhood of *any* point of F, and not merely so in the neighborhood of a generic point of the curves of discontinuity as the definition simply asserts. As a corollary when F is birationally transformed into a surface with ordinary singularities F', an integral of the second kind J of F becomes one of the second kind J' of F'. For if J_1 is the improper integral corresponding to a point M transformed into a point M' or into a curve M'' (fundamental curve) of F' and J_1' the integral into which J_1 is changed, the difference $J - J_1 = J' - J_1'$ being finite in the vicinity of M on F, will also remain finite in that of M' or of a generic point of M'' on F and as J_1' is improper of the second kind for F' our assertion is justified.

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* Picard-Simart, vol. II, p. 17.

† Picard-Simart, vol. II, p. 186.

Remark. Integrals of the second kind belonging to a surface F with arbitrary singularities may be defined as those which become of the second kind when F is transformed into a surface with ordinary singularities.

33. With R_2 integrals of the second kind of type (20) we can always form a linear combination without periods and therefore improper. Hence the number ρ_0 of unequivalent integrals of the second kind is given by

$$\rho_0 = R_2 - \rho = I + 4q_1 - \rho + 2; \qquad (\rho \text{ integer} \geqq 1),$$

which is Picard's classical formula.* In the next section the number ρ will be identified with the one that he has designated by the same letter. According to No. 28 ρ_0 remains invariant under birational transformations. As is well known this is also the case for q but not for I and therefore not for R_2 or ρ. In the customary terminology ρ_0, q are absolute invariants and ρ, I, R_2, are relative invariants.

§ 5. *Integrals of total differentials.*

34. An integral of total differentials is an integral such as

$$\int U dy + V dx, \tag{21}$$

where U, V, are rational functions of F satisfying the condition of integrability of Poincaré

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$$

thanks to which (21) is a point-function on the surface. The integral has *periods* obtained by taking for paths of integration linear cycles of F. If the cycle reduces to a small circuit around a curve of discontinuity the period is called *logarithmic*, otherwise it is called *cyclic*. When the small circuit is displaced without intersecting the curves of discontinuity the logarithmic period is unchanged so that it really belongs to the curve of discontinuity considered, also called *logarithmic curve*. As any linear cycle is homologous to one in a given plane section, the periods of (21) are also those of

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* See Picard-Simart, vol. II, p. 497. This formula holds even for a ruled or a Steiner surface. Indeed the whole argument holds with only slight modifications if we replace everywhere $|H|$ by $|2H|$, this being also true for the next section. As a matter of fact for a ruled surface

$$N = m, \qquad R = 2p, \qquad \rho = 2 \text{ (Severi)}, \qquad \therefore \rho_0 = 0.$$

That is, a ruled surface has no proper integrals of the second kind. This holds also for a Steiner surface since it is rational.

$$\int U dy, \qquad \int V dx \tag{22}$$

belonging to H_x, H_y respectively.

Integrals of total differentials have been divided into three kinds on the same basis as abelian integrals. The first kind includes integrals finite everywhere; the second, integrals without logarithmic curves; and the third, integrals possessing some.

Let $C_1, C_2, \cdots, C_\nu$ be irreducible logarithmic curves of (21), m_i, μ_i the order of C_i and the corresponding logarithmic period. The integrals (22) will have m_i logarithmic points with a period of μ_i for each, therefore $\sum_{i=1}^{\nu} m_i\mu_i = 0$. This shows that an integral of the third kind has at least *two* logarithmic curves. In point of fact we shall show in the next section that $\rho + 1$ is precisely the least number of logarithmic curves of such an integral.

The study of integrals of the first kind forms one of the most interesting chapters of algebraic geometry, but as it is unrelated to the considerations of this paper we shall omit it here.

§ 6. *Integrals of total differentials of the second kind.*

35. In the notations of No. 25 we may consider the system of equations in the unknowns c,

$$\sum_{h=1}^{2p+m-1} c_h\Omega_{hj}(y) = 0; \qquad \sum_{h=1}^{2p+m-1} c_h\omega_{hk}(y) = d_k;$$
$$(j = 1, 2, \cdots, N;\ k = 1, 2, \cdots, 2q),$$

where the (d)'s are arbitrary constants not all zero. It may be shown here again that there is a unique solution composed of rational functions of y. The integral

$$\int V dx \qquad \left(V = \sum \frac{c_h Q_h(x, y, z)}{F_z'}\right),$$

has only the (d)'s for periods, hence

$$\frac{\partial}{\partial y}\int V dx \tag{23}$$

is a rational function on H_y. Reasoning as before, we may show that there exists a rational function $U(x, y, z)$ such that

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y},$$

so that

$$\int U dy + V dx$$

is an integral of total differentials. It may have logarithmic curves belonging to $\{H_y\}$ but these may be eliminated, as we have had occasion to show, by subtracting integrals

$$\int \frac{\alpha dy}{y - a},$$

which do not affect the periods with respect to the linear cycles of F, periods equal to those of (25) that is to the (d)'s. The integral finally obtained is said to be *of the second kind.* Obviously we may have $2q$ such integrals with no linear combination deprived of periods, that is with no linear combination equal to a rational function, which is expressed by the statement that *there are $2q$ linearly independent integrals of total differentials of the second kind.* Let us mention that *the number of linearly independent integrals of total differentials of the first kind is equal to q.*

§ 7. *Integrals of total differentials of the third kind and their relation to ρ.*

36. We have seen that there are $R_2 - 1$ distinct integrals of type (19) with non-zero periods. Hence if $\rho = R_2 - \rho_0 > 1$, $\rho - 1$ of these integrals must be improper, that is there must be $\rho - 1$ distinct improper integrals of type (19) with non-zero periods. When (19) is improper

$$\frac{Q(x, y, z)}{\varphi(y)F_z'} = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}$$

and according to Nos. 30, 31 and with similar notations U, V are both of the form

$$\frac{A(x, y, z)}{\prod_i [g_i(x, y)]^{\alpha_i} \cdot F_z'}.$$

The logarithmic periods of

(24) $$- \int V dx$$

relatively to the points where H_y meets D_i, irreducible curve of infinity of U, V, not contained in $\{H_y\}$, are functions of y alone. But their derivatives with respect to y are zero, hence they have a constant value μ_i, the same for all points on D_i. Now if (19) has no periods

$$\frac{Q(x, y, z)}{\varphi(y)F_z'} = \frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y}$$

where U_1, V_1 are both of the form

(24) $$\frac{A(x, y, z)}{\psi(y) \cdot F_z'}.$$

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* See Picard-Simart, vol. I, p. 100.

Comparing we obtain

$$\frac{\partial(U - U_1)}{\partial x} + \frac{\partial(V - V_1)}{\partial y} = 0,$$

hence

$$\int (U - U_1)dy - (V - V_1)dx \tag{25}$$

is of total differentials. Let H_{y_k} (y_k finite), be a curve of discontinuity of (25), $2\pi i \alpha_k$ the corresponding logarithmic period. *The integral*

$$\int \left(U - U_1 - \sum \frac{\alpha_k}{y - y_k}\right) dy - (V - V_1)dx$$

has no other logarithmic curves than H_∞ and the (D's), the period relatively to D_i being μ_i.

37. Conversely let us assume the existence of an integral

$$\int U_2 dy - V_2 dx, \qquad \frac{\partial U_2}{\partial x} + \frac{\partial V_2}{\partial y} = 0$$

such as we have just described. As we may write

$$\frac{Q(x, y, z)}{\varphi(y)F_z'} = \frac{\partial(U - U_2)}{\partial x} + \frac{\partial(V - V_2)}{\partial y}$$

the integral

$$- \int (V - V_2)dx$$

behaves like a rational function everywhere on H_y except perhaps at infinity. As its points of discontinuity are determined rationally in terms of y, there exists a rational function $R(x, y, z)$ such that

$$V_1 = V - V_2 - \frac{\partial R}{\partial x}$$

is of the form

$$\frac{A(x, y, z)}{\varphi(y)F_z'}.$$

Then owing to the identity

$$\frac{Q(x, y, z)}{\varphi(y)F_z'} = \frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y}, \qquad U_1 = U - U_2 + \frac{\partial R}{\partial y},$$

U_1 is of the same form as V_1 and finally (19) has no periods. Hence, *in order that the double integral* (19) *be without periods it is necessary and sufficient that there exist an integral of total differentials with H_∞ and the (D)'s for logarithmic curves and a period for D_i equal to the corresponding period of the abelian integral* (23) *for any point on this curve.*

38. We now propose to form an abelian integral belonging to H_y of type

$$\int \frac{A(x, z)dx}{g_i(x, y)F_z'}$$

(A indeterminate polynomial adjoint to the curve), with no other discontinuities at finite distance than the points on D_i the corresponding logarithmic periods being all equal to $+1$. The relations which this imposes upon the coefficients of A, assumed of conveniently high degree are certainly compatible and moreover rational with respect to y, hence the integral is of the form

$$\int \frac{A(x, y, z)dx}{\psi(y)g_i(x, y)F_z'} = -\int V_i'dx,$$

where A is now an adjoint polynomial vanishing on the curve other than D_i projected on $g_i(x, y) = 0$. The abelian integral

$$-\frac{\partial}{\partial y}\int V_i'dx$$

behaves like a rational function at finite distance, hence as above there exists a rational function U_i' such that

$$\frac{P_i'(x, y, z)}{\chi(y)F_z'} = \frac{\partial U_i'}{\partial x} + \frac{\partial V_i'}{\partial y}$$

(P_i' adjoint polynomial, χ polynomial in y) this identity itself showing that U_i' is of the same type as V_i'.

For the sequel it is necessary to operate a further reduction so as to remove $\chi(y)$ concerning whose roots nothing is known. From the discussion of No. 27 it follows that we can find two functions U_i'', V_i'' of type

$$\frac{B(x, y, z)}{\theta(x)\theta_1(y)F_z'}, \quad (\theta,\ \theta_1,\ B, \text{ polynomials, the last adjoint})$$

such that

$$\frac{P_i'(x, y, z)}{\chi(y)F_z'} - \frac{\partial U_i''}{\partial x} - \frac{\partial V_i''}{\partial y} = \frac{P_i(x, y, z)}{F_z'} = \frac{\partial U_i}{\partial x} + \frac{\partial V_i}{\partial y};$$

$$(U_i = U_i' - U_i'',\ V_i = V_i' - V_i'').$$

The abelian integral

$$(26) \qquad -\int V_i dx$$

may have logarithmic periods with respect to points on curves H_x but to the points on a given H_{x_k} will correspond the same constant period say $2\pi i\beta_k$. The functions

$$V_i, \qquad V_i - \sum\frac{\beta_k}{x - x_k},$$

have the same partial derivatives with respect to y, hence the first may be replaced by the second in the identity which defines it, and the corresponding integral (26) will then have nowhere logarithmic periods with respect to points on the curves H_{x_k}. We will thus be able to write

$$\frac{P_i(x, y, z)}{F_z'} = \frac{\partial U_i}{\partial x} + \frac{\partial V_i}{\partial y},$$

the integral (26) having now no other logarithmic singularities at finite distance than the points on D_i with corresponding periods equal to $+1$.

39. Let us now return to the consideration of improper integrals of form (19). We have

$$\int\int\left[\frac{Q(x, y, z)}{\varphi(y)F_z'} - \sum \mu_i \frac{P_i(x, y, z)}{F_z'}\right] dxdy = \int\int\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) dxdy.$$

The double integral at the right is again of form (19) but this time

$$-\int V dx$$

has no logarithmic singularities at finite distance and therefore the double integral has no periods. We conclude from this that the number $\rho - 1$ of distinct improper integrals of form (19) admitting of no linear combination without periods is the same as when only integrals

$$J_i = \int\int \frac{P_i(x, y, z)dxdy}{F_z'} = \int\int\left(\frac{\partial U_i}{\partial x} + \frac{\partial V_i}{\partial y}\right) dxdy$$

are considered. But if $\sum_{i=1}^{\rho'} \mu_i J_i$ is without periods there exists an integral of total differentials with $H_\infty, D_1, D_2, \cdots, D_{\rho'}$, for sole logarithmic curves, and a period μ_i with respect to D_i and conversely. Hence the number $\rho - 1$ of distinct (J)'s of which no linear combination is without periods is equal to the maximum number of curves of F which together with H_∞ may not form the logarithmic curves of an integral of total differentials.

Let $D_1, D_2, \cdots, D_{\rho+1}$ be any $\rho + 1$ curves of F. There are two integrals of total differentials, I_1, I_2 with the respective logarithmic curves $H_\infty, D_1, D_2, \cdots, D_\rho$ and $H_\infty, D_1, \cdots, D_{\rho-1}, D_{\rho+1}$, and periods α_1, α_2 relatively to H_∞. One of the two integrals of total differentials $\alpha_2 I_1 - \alpha_1 I_2$, I_1 (if $\alpha_1 = 0$) has $D_1, D_2, \cdots, D_{\rho+1}$ for sole logarithmic curves. Therefore *the number ρ is the maximum number of algebraic curves which may not be logarithmic curves of an integral of total differentials*, which is the very interpretation given for this number by Picard.

40. Let us recall briefly how Picard arrived at the number ρ.[*] Starting with an algebraic curve D_i he first formed an abelian integral of the third

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[*] Picard-Simart, vol. II, p. 231, Ch. 9.

kind belonging to H_y

$$J_i = \int V_i dx$$

having the points on D_i for sole logarithmic points the corresponding periods being all $+1$. Let $\omega_{i,1}, \omega_{i,2}, \cdots, \omega_{i,\,2p+m-1}$, be a fundamental system of periods of J_i. When y turns around one of the critical points $\omega_{i\mu}$ becomes

$$\omega'_{i\mu} = \sum m_{\mu\nu}\omega_{i\nu} + m_\mu,$$

the integers $m_{\mu\nu}$, m_μ being independent of J_i. From this readily follows that by taking a sufficiently high number of these integrals we can form a linear combination with coefficients $b_i(y)$ rational in y, as

$$J = \sum b_i(y) J_i = \int V dx$$

with all periods constant. Hence as in No. 35 there exists an integral of total differentials

$$\int U dy + V dx,$$

with the (D)'s and the curve at infinity for sole logarithmic curves. This shows that there is a maximum ρ of the number of curves which may not become logarithmic curves of an integral of total differentials which is precisely the definition found for ρ in No. 39.

§ 8. *Summary of further developments concerning the number ρ.*

41. Only second in importance to Picard's work must be set Severi's discovery of the relation between the preceding theory and the distribution of algebraic systems of curves on the surface.[*] The writer may also mention his own contributions which connect the Picard-Severi theory to the topology.[†] We propose to dwell rapidly on these various developments.

By definition a continuous complete system $\{C\}$ of curves of F is said to be the sum of systems $\{A\}$, $\{B\}$, if there is a C composed of an A and a B. The system is perfectly defined and we write

$$\{C\} = \{A + B\}, \qquad \text{or} \qquad C = A + B.$$

From this to consideration of systems $\{A - B\}$, or more generally $\sum \lambda_i C_i$ where the (λ)'s are integers is but a step. We write $\sum \lambda_i C_i = 0$ and say that the curves are *algebraically dependent* if there is a C such that

$$\{C + \sum \lambda_i C_i\} = \{C\}.$$

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[*] Sulla base per la totalita delle curve . . . , Math. Ann., vol. 62 (1906), pp. 194–225. La base minima pour la totalité des courbes . . . , Ann. Ec. Norm., ser. 3, vol. 25 (1908). See also two very interesting Memoires by Poincaré, Ann. Ec. Norm. Sup., ser. 3, vol. 27 (1910), pp. 55–108. Berliner Sitzungsb., vol. 10 (1911), pp. 28–55, where results closely related to Severi's are obtained by means of Abel's theorem.

[†] Rendiconti dei Lincei, loc. cit.

Let $[AB]$, $[A^2]$, denote the number of points common to generic curves A, B, or to two curves of $\{A\}$, respectively. Severi has established that: *In order that* $\lambda A = \lambda B$ (λ some integer) *it is necessary and sufficient that*

$$[A^2] = [AB] = [B^2].$$

By means of this criterium he showed finally that: *The curves* $C_1, C_2, \cdots, C_n$, *are algebraically dependent if and only if they are the logarithmic curves of an integral of total differentials of the third kind.* This is his fundamental result. From this follows that there can be found ρ independent curves $C_1, C_2, \cdots, C_\rho$ such that for any other C there is a relation

$$\lambda C = \sum_{i=1}^{\rho} \lambda_i C_i.$$

The curves C_i form a so-called *base* for all the curves of the surface. Thus on a ruled surface $\rho = 2$ a base being constituted by a generator and a plane section, on a quadric again $\rho = 2$ and two generators not of the same system form a base.

42. We have already mentioned that an algebraic curve forms a two-dimensional cycle. We will call *algebraic* cycle a cycle homologous to one of this nature. It may be shown, as is only partly done in the writer's Lincei note, that *a cycle is algebraic if and only if double integrals of the first kind have no periods with respect to it.* Hence ρ *is the number of distinct cycles with respect to which these integrals have no periods.* From this may be deduced the following fundamental result: *To a homology between algebraic cycles corresponds a relation of dependence between the curves and vice versa.* These theorems find numerous applications of which we may mention one—the first correct proof of a theorem stated by Nöther according to which the most general surface of order > 3 possesses no other curves than complete intersections.